

# DYNAMICAL PERCOLATION MODELS OF PHASE TRANSITIONS IN THE CORTEX

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**Abstract**— We propose a series of models based on statistical processes that resemble behavior of the cortex. At one extreme we propose models of individual and collections of neurons that have a high degree of biological plausibility. At the other extreme are simpler models that can be mathematically analyzed rigorously. We aim to construct models that both encompass the complex behavior of biological systems while at the same time are mathematically tractable.

## I. INTRODUCTION

Contrary to the presently used digital computer memories where information is encoded in the form of a given string of binary digits we propose a novel approach, in which the encoding is embodied in oscillations of the activity of the memory nodes. The approach is strongly biologically motivated and based on the observation that humans and animals can solve difficult identification tasks fast and with high accuracy based on a few examples [13], [15], [21].

Information processing in the proposed dynamical memories is closely related to percolation phenomena [15], [8], [20]. For about forty years now, percolation theory has been an active area of research at the interface of probability theory, combinatorics and physics [17]. Interest in various aspects of standard percolation remains high, including estimates of critical probabilities [6], [5]. In the last decade and a half, more and more variants of the standard percolation models have been studied. In particular, there has been much work on the family of cellular automata known as bootstrap percolation [3], [11], [2], [23], [22], [16], [10], and many others). Computer experiments performed by physicists have suggested interesting non-trivial large-scale behavior and many deep mathematical results have been proved about a number of models.

Phase transitions in various physical and biological systems are also described by [19], [26], [24]. We apply the solid mathematical theory of percolation in lattices to lay down the foundations of dynamical memories and the related phase transitions. When an input

pattern is presented to the model, the aperiodic oscillations undergo a phase transition and the trajectory is switched to a localized memory wing that corresponds to the identified class label of the actual input pattern. When the input pattern is removed, the perturbation that kicked the trajectory into the local memory wing ceases and the system returns to the high-dimensional basal state [14].

Mathematically such a property has been described in random graphs, where the connectivity density is an order parameter that can induce state transitions [12], [9], [20]. Accordingly, the memory and classification process can be described as percolation phenomenon through the neurophil medium [15]. The problem is related to small-world phenomena [27], [18], [25]. Recently, certain empirical observations about scaling properties of large graphs have been proven rigorously based on graph-theoretical arguments [7].

In this work, we extend percolation theory to interpret the behavior of distributed dynamical memories. We study a wide variety of infection and recovery functions in bootstrap percolation, and thereby model a large family of evolutions and phase transitions in layers of excitatory and inhibitory nodes. We perform numerous simulations and we underpin the conclusions with precise mathematical results.

## II. RANDOM GRAPHS AND MODELS OF PHASE TRANSITIONS

### A. Critical Phenomena

Percolation theory deals with large scale properties of certain types of random graphs, often built from lattices such as  $Z^n$ . In the archetypal percolation problem the vertices (sites) are the points of a lattice such as  $Z^n$  with edges or bonds joining neighboring sites. In site percolation, sites are open independently with probability  $p$  and one wishes to answer questions about the size of the connected components formed by these open sites. In particular, do infinite connected clusters of open sites exist? Similar questions can also be asked about bond percolation, where the bonds are

chosen to be open with a certain probability. There are many variants of these problems. For example, in oriented percolation one asks for infinite paths of connected open sites that travel at each step only in certain directions. Many percolation problems exhibit phase transitions, where for  $p$  less than some critical probability  $p_{crit}$  only finite clusters exist, and for  $p > p_{crit}$  infinite clusters almost surely exist.

## B. Bootstrap Percolation

In the framework of this research, percolation theory and its generalization will be used for the interpretation of the behavior of chaotic dynamical memories. A rather general form of bootstrap percolation can be defined as follows. Let  $G$  be a graph,  $0 < p < 1$  a probability, and  $k > 1$  a natural number. The vertices of  $G$  are called sites, and the edges are bonds; two sites are neighboring if they are joined by a bond. A site is either occupied or unoccupied. Frequently, occupied sites are referred to as active and unoccupied ones as inactive. In the bootstrap percolation on  $G$  with parameters  $p$  and  $k$ , the sites are independently occupied at time 0, with probability  $p$ , and at each time step every empty site with at least  $k$  occupied neighbors becomes occupied. The graph  $G$  is often lattice-like; in fact,  $G$  is frequently taken to be an infinite lattice  $Z^d$ . Among many other properties of this percolation, one is interested in the critical value of  $p$ , above which every site eventually becomes occupied (with probability 1). For example, Schonman [23] proved that for the cubic lattice  $G = Z^3$  and  $k = 3$ , the critical probability is 0.

It is perhaps even more interesting to study bootstrap percolation on finite graphs (finite volume models). For example, instead of  $G = Z^3$  we may consider the 3-dimensional torus graph  $G = Z_n^3$  on  $n^3$  vertices, i.e., the product of three cycles, each of length  $n$ . Note that the graph  $Z_n^3$  is homogeneous: every vertex has degree 6 (for  $n > 2$ ) and any vertex can be mapped to any other vertex by an automorphism of  $Z_n^3$ . Concerning bootstrap percolation on  $(G, p, k) = (Z_n^3, p, 3)$ , [10] proved the conjecture of Adler et al. [1] extending the above result of Schonman, that the critical probability is of the order  $1/\log \log n$ .

## III. RANDOM CELLULAR AUTOMATA AS GENERALIZED BOOTSTRAP PERCOLATION

### A. Definitions

We study a novel class of percolation processes. We shall extend the rules governing the evolution of the system in two important ways: (i) we shall allow long-range infections to take place; (ii) we shall allow infected sites to recover at a certain rate. Clearly, the family of the generalized bootstrap percolations is much richer than the family of bootstrap percolations.

In particular, with a suitable choice of the infection and recovery functions, we may achieve that the system hovers around a certain density of infected sites.

In our model, various phase transition phenomena may occur. For example, we expect to find fairly simple functions  $j$  and  $r$  for which much of the time the system is in  $k$  prescribed states, i.e., at most times  $t$ , the density  $|S_t|/n^2$  is close to  $k$  prescribed values  $p_1, \dots, p_k$ ;  $0 < p_1 < \dots < p_k < 1$ . Yet another choice of  $j$  and  $r$  could result in the system oscillating in a prescribed range. By identifying conditions of such oscillatory behavior in the bootstrap percolation model, a firm mathematical basis of the arising phase transition and consequent memory effects in dynamical memories will be established. This, in turn, will help to select optimum parameters (learning) and robust performance of the classification using few examples.

Let us describe one of the many possibilities on the 2-dimensional discrete torus  $Z_n^2$ . For  $x \in Z_n^2$ , let  $\|x\| = |x_1| + |x_2|$ , where  $x = (x_1, x_2)$  and each  $x_i$  is chosen to be at most  $n/2$  in modulus. Let  $1 \leq a < n/2$  and  $A = \{x \in Z_n^2 : \|x\| \leq a\}$ . The fate of each site  $x$  will be influenced by the sites in the domain  $x + A$ . Introduce here  $j$  as the arousal function and  $r$  as the depression function. At time 0, the sites are active with probability  $p$ . At each time step  $t$ , every site is updated simultaneously according to the following rule. Write  $S_t$  for the set of active sites at time  $t$  and set

$$B = [S_t \cap (A + x)] - x = (S_t - x) \cap A.$$

Here  $(A + x)$  is also called the  $a$ -ball around  $x$ . We should think of  $A + x$  as the set of occupied sites near enough to  $x$  to influence  $x$ . If  $x \notin S_t$  (i.e.,  $0 \notin B$ ), then  $x$  will become active with probability  $j(B)$ , and if  $x \in S_t$  (i.e.,  $0 \in B$ ) then  $x$  will be deactivated with probability  $r(B)$ .

In the case when no cell can change from active to inactive ( $r(B)=0$  for all  $B$ ) we recover the bootstrap percolation models described above. The standard bootstrap percolation on  $Z_n^2$  with probability  $p$  and  $l = 2$  is a trivial special case: take  $a = 1$  and for  $B \subseteq A$ , let  $r(B) = 0$  and

$$j(B) = \{1, |A \cap B| > 1; \text{ otherwise}\}$$

. Clearly however, the family of random cellular automata is much richer than the family of bootstrap percolations. In particular, if all the probabilities are 0 or 1 we recover the theory of deterministic cellular automata, such as that of Conway's Game of Life. These models are known to be capable of extremely complex behavior. On the other hand, if we choose  $j(B), r(B) \neq 0, 1$  for all  $B$  then there is no need for an initial probability  $p$ , and with a suitable choice of the arousal and depression functions, we may achieve that

the system hovers around a certain density of active sites.

## B. Phase Transitions in the Local Model

As a simple example of models with local neighborhoods, take  $a = 1$  and

$$j(B) = \{p, \text{if } |A \cap B| < 3; 1 - p \text{ otherwise}\},$$

$$r(B) = \{p, \text{if } |A \cap B| > 2; 1 - p \text{ otherwise}\}.$$

In this case the site is active with probability  $p$  if at most 2 of the four neighbors and the site itself are active at time  $t - 1$ . Otherwise the site is inactive with this probability. As we vary  $p$  this model shows a phase transition similar to that seen in the Ising model. On an infinite grid and for small  $p$  there are two stable states, one with low density  $\rho$  and one with high density  $1 - \rho$ . As  $p$  increases so does  $\rho$ , until at some critical value of  $p$  the density becomes 0.5. On a finite grid the probability distribution of active cells is highly bimodal for small  $p$ . There is of course a small probability of the system switching between high and low density states but in practice this is very rare if the grid is large. For large  $p$  the distribution of the number of active sites becomes unimodal. For  $p$  close to the critical probability one sees large regions of high density and large regions of low density on the grid evolving with time. Also, the variance of the density on a finite grid becomes very large as  $p$  approaches the critical probability [4].

## C. The Case of Random Neighborhoods

A simpler model can be obtained by instead of taking the four closest neighbors one takes four random points on the grid, chosen independently and uniformly at each site and at each time step. This model is a mean field version of the previous one and has the advantage that it can be analyzed rigorously. The behavior is qualitatively similar except that there is no topological structure on the grid, so no well defined regions as above. For this model the phase transition can be shown to occur at  $p = 7/30 = 0.233$ . One can calculate the exact probability distribution for the number of active sites using linear algebra. This value was calculated exactly by computer for the  $N=400$  grid. For the dynamic Ising-like model described above the critical probability appears to be somewhat lower at about  $p = 0.14$ .

An even closer connection to the Ising model can be given if we take the single update model with  $r(B)$  and  $j(B)$  proportional to an exponential in the number of neighbors that are in the same state as the site. In other words, take

$$j(B) = \epsilon \exp(\lambda |B|),$$

$$r(B) = \epsilon \exp(\lambda |A \setminus B|),$$

with small  $\epsilon$ . In this case the probability distribution of the configurations tends to that of the Ising model with parameter  $\lambda$ , as  $\epsilon \rightarrow 0$  and  $\epsilon t \rightarrow \infty$ .

If the functions  $r(B)$  and  $j(B)$  depend on the shape of  $B$  rather than just the number of active sites then even more complex behavior is possible. In the simplest case when  $a = 1$  there are 32 parameters, one for each possible state of the four neighbors and the site itself at time  $t - 1$ . A very wide range of behaviors has been observed with these models, including several previously studied models. Many types of behavior do not fit neatly into the patterns described above. For example there are models where one phase propagates through the grid at a given velocity and another phase of the same model is stationary.

## IV. Connection between generalized percolation and neurodynamics

We have seen cases where there are phase transitions between high-dimensional, aperiodic and quasi-periodic behavior, or between static and periodic behavior analogous to the Ising-like models mentioned above. The model was sensitive to the parameters and a slight change in the parameters can switch the model between fixed point and limit cycle regimes with random fluctuations. Moreover, going from the cycle to the fixed point gives a rapidly decaying sine wave. These are of particular interest since this is the regime expected to arise in KII systems with excitatory and inhibitory layers. By identifying conditions of such oscillatory behavior in the random automata model, a firm mathematical basis of the arising phase transition and consequent memory effects in dynamical memories will be established. This, in turn, will help to select optimum parameters (learning) and robust performance of the classification using few examples.

We proposed a series of models based on statistical processes that resemble behavior of the cortex. In particular we observed complex dynamics. We plan to use this approach, with appropriate habituation and hebbian learning rules, to solve difficult problems of pattern recognition. At one extreme we propose models of individual and collections of neurons that have a high degree of biological plausibility. At the other extreme are simpler models that can be mathematically analyzed rigorously. We aim to construct models that both encompass the complex behavior of biological systems while at the same time are mathematically tractable.

In a more advanced model simulating the interaction of a pair of excitatory-inhibitory layers, we have observed both quasi-periodic as well as aperiodic behavior that is more complex than the simple periodic or quasi-periodic oscillatory behavior using differential equations. Also this model can be modeled with arbitrarily high accuracy by a random cellular automata

which depends on the states of the neighbors at a finite number of previous times.

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